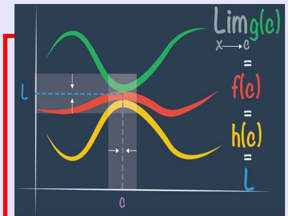
Math 261

Fall 2022

Lecture 7



Squeeze Theorm:

If $f(x) \le g(x) \le h(x)$ when x is near a, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} g(x) = L$

Ex: Find $\lim_{x \to 0} x^4 \sin \frac{1}{x}$

Recall

$-1 \le \sin A \le 1$

So $-1 \le \sin \frac{1}{x} \le 1$

but we need $x^4 \sin \frac{1}{x}$

Multiply by x^4.

Since $x^4 \ge 0$

$$x^4(-1) \le x^4 \sin \frac{1}{x} \le x^4(1)$$

$$-x^4 \le x^4 \sin \frac{1}{x} \le x^4$$

Now $\lim_{x \to 0} (-x^4) = 0$

$\lim_{x \to 0} x^4 = 0$

by S.T.,

$$\lim_{x \to 0} x^4 \sin \frac{1}{x} = \boxed{0}$$

Suppose $2x \le g(x) \le x^4 - x^2 + 2$ for all x,

$\forall$

Find $\lim\limits_{x \to 1} g(x)$

$\lim\limits_{x \to 1} 2x = 2$

$\lim\limits_{x \to 1} (x^4 - x^2 + 2) = 2$

So by S.T.,

$\lim\limits_{x \to 1} g(x) = \boxed{2}$

Suppose $4x - 9 \le f(x) \le x^2 - 4x + 7 \quad \forall\, x \ge 0$,

Find $\lim\limits_{x \to 4} f(x)$

$\lim\limits_{x \to 4} (4x - 9) = 4(4) - 9 = \boxed{7}$

$\lim\limits_{x \to 4} (x^2 - 4x + 7) = 4^2 - 4(4) + 7 = \boxed{7}$

By S.T.

$\lim\limits_{x \to 4} f(x) = \boxed{7}$

Consider $f(x) = \begin{cases} x^2+1 & \text{if } x<1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$

$x^2+1 \;\Big|\; (x-2)^2$

$\longrightarrow 1 \longleftarrow$

1) $f(1) = (1-2)^2 = \boxed{1}$

2) $\lim\limits_{x \to 1^-} f(x) = 1^2 + 1 = \boxed{2}$

3) $\lim\limits_{x \to 1^+} f(x) = (1-2)^2 = \boxed{1}$

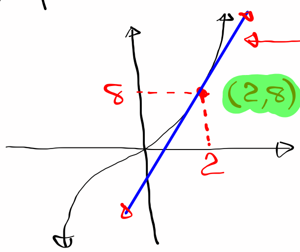
4) $\lim\limits_{x \to 1} f(x) =$ D.N.E.

$\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

5) Is $f(x)$ cont. at $x=1$? Explain

NO, $\underbrace{\lim\limits_{x \to 1} f(x)}_{\text{D.N.E.}} = \underbrace{f(1)}_{1}$

Find equation of the tangent line to the graph of $f(x) = x^3$ at $x=2$.



$m = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(x+h)^3 - x^3}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$

$= \lim\limits_{h \to 0} (3x^2 + 3x\underbrace{h}_{\to 0} + \underbrace{h^2}_{\to 0})$

$= 3x^2$

at $x = 2$

$m_{\text{tan. line}} = 3(2)^2 = \boxed{12}$

$y - y_1 = m(x - x_1)$

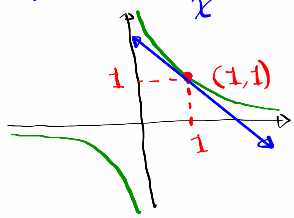
$y - 8 = 12(x - 2)$

$y = 12x - 24 + 8$

$\boxed{y = 12x - 16}$

Find slope of the tangent line at $x=1$

for $f(x) = \frac{1}{x}$.



$$m_{\text{tan. line}} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

LCD $= x(x+h)$

$$= \lim_{h \to 0} \frac{x\cancel{(x+h)} \cdot \frac{1}{\cancel{x+h}} - \cancel{x}(x+h) \cdot \frac{1}{\cancel{x}}}{x(x+h) \cdot h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h\,x(x+h)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\,x(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+\cancel{h})}$$

$$= \boxed{\frac{-1}{x^2}}$$

At $x=1$

$$m_{\text{tan. line}} = \frac{-1}{(1)^2}$$

$$= \boxed{-1}$$

Google Normal line

Read about it.

Precise def. of limit:

If $\lim_{x \to a} f(x) = L$

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

Find a relationship between ε and δ for $\lim_{x \to 2} x^2 = 4$.

1) Verify the limit.

$\lim_{x \to 2} x^2 = 2^2 = 4 \checkmark$

2) Identify $f(x)$, a, and L.

$f(x) = x^2$, $a = 2$, $L = 4$

3) By precise def. of limit,
for $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 - 4| < \varepsilon$ whenever $|x - 2| < \delta$

$|AB| = |A| \cdot |B|$

$|(x+2)(x-2)| < \varepsilon$

$\underbrace{|x+2|}_{\text{Bound}} \underbrace{|x-2|}_{\text{Keep}} < \varepsilon \longrightarrow 5|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{5}$

Let's say we want δ to be no more than 1.

$|x-2| \leq 1$

$-1 \leq x - 2 \leq 1$

Add 2

$1 \leq x \leq 3$

Add 2 again

$3 \leq x + 2 \leq 5$

We pick $\delta = \frac{\varepsilon}{5}$

$\varepsilon = 5 \to \delta = 1$

$\varepsilon = 2 \to \delta = .4$

$\varepsilon = 15 \to \delta = 1$

$\boxed{\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}}$

Class QZ 2 — Portrait Style

1) Find $\lim_{x \to 3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{3^2 - 9}{3^2 + 2(3) - 3} = \frac{0}{12} = \boxed{0}$

2) Find $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{4} - 2}{4 - 4} = \frac{0}{0}$ I.F.

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x-4)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{x - 4}{(x-4)(\sqrt{x} + 2)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \boxed{\frac{1}{4}}$$